ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Math. Anal. Appl. 312 (2005) 564–575

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Some classes of analytic functions involving Noor integral operator ☆

J. Patel [a], N.E. Cho [b,*]

[a] *Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar-751004, India*

[b] *Department of Applied Mathematics, Pukyong National University, Pusan 608-737, South Korea*

Received 21 February 2005

Available online 12 April 2005

Submitted by M.A. Noor

Abstract

The object of the present paper is to investigate some inclusion properties of certain subclasses of analytic functions defined by using the Noor integral operator. The integral preserving properties in connection with the operator are also considered. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

Keywords: p-Valent; Noor integral operator; Differential subordination

1. Introduction

Let $\mathcal{A}_p$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad \bigl(p \in \mathbb{N} = \{1, 2, 3, \ldots\}\bigr), \tag{1.1}$$

☆ The present investigation was supported by the financial assistance received from UGC, New Delhi under DRS programme, and the second author was supported by Pukyong National University Research Foundation Grant in 2004.

* Corresponding author.

E-mail addresses: jpatelmath@sify.com (J. Patel), necho@pknu.ac.kr (N.E. Cho).

0022-247X/$ – see front matter

doi:10.1016/j.jmaa.2005.03.047

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C}\colon |z| < 1\}$. We denote $\mathcal{A}_1 = \mathcal{A}$. If f and g are analytic in $\mathcal{U}$, we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function ω in $\mathcal{U}$ such that $f(z) = g(\omega(z))$. Let $\mathcal{S}_p^*(\eta)$ and $\mathcal{K}_p(\eta)$ be the subclasses of $\mathcal{A}_p$ consisting of all analytic functions which are, respectively, p-valently starlike and p-valently convex of order η $(0 \leqslant \eta < p)$ in $\mathcal{U}$. We denote $\mathcal{S}_1^*(\eta) \equiv \mathcal{S}^*(\eta)$ and $\mathcal{K}_1(\eta) \equiv \mathcal{K}(\eta)$, the usual classes of starlike and convex functions of order η $(0 \leqslant \eta < 1)$ in $\mathcal{U}$ [9].

For real or complex numbers a, b, c $(c \neq 0, -1, -2, \ldots)$, the hypergeometric series is defined by

$$ {}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots. \tag{1.2}$$

We note that the series in (1.2) converges absolutely for $z \in \mathcal{U}$ and hence represents an analytic function in $\mathcal{U}$ [17].

For $f \in \mathcal{A}_p$, we denote by $\mathcal{D}^{\delta+p-1}\colon \mathcal{A}_p \to \mathcal{A}_p$ the operator defined by

$$\mathcal{D}^{\delta+p-1} f(z) = \frac{z^p}{(1-z)^{\delta+p}} * f(z) \quad (\delta > -p) \tag{1.3}$$

or, equivalently, by

$$\mathcal{D}^{n+p-1} f(z) = \frac{z^p(z^{n-1}f(z))^{(n+p-1)}}{(n+p-1)!},$$

where n is any integer greater than $-p$ and the symbol $(*)$ stands for the Hadamard product (or convolution). If $f(z)$ is given by (1.1), then from (1.3) it follows that

$$\mathcal{D}^{n+p-1} f(z) = z^p + \sum_{k=p+1}^{\infty} \binom{n+k-1}{k-p} a_k z^k \quad (p \in \mathbb{N};\ n > -p).$$

The symbol $\mathcal{D}^{n+p-1}$ when $p = 1$ was introduced by Ruscheweyh [15] and the symbol $\mathcal{D}^{n+p-1}$ was introduced by Goel and Sohi [4]. This symbol is called as the Ruscheweyh derivative of $(n+p-1)$th order.

Recently, analogous to $\mathcal{D}^{n+p-1} f$, Liu and Noor [6] introduced an integral operator $\mathcal{I}_{n,p}\colon \mathcal{A}_p \to \mathcal{A}_p$ as follows:

Let $f_{n,p}(z) = z^p/(1-z)^{n+p}$ $(n > -p)$, and let $f_{n,p}^{(\dagger)}(z)$ be defined such that

$$f_{n,p}(z) * f_{n,p}^{(\dagger)}(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.4}$$

Then

$$\mathcal{I}_{n,p} f(z) = f_{n,p}^{(\dagger)}(z) * f(z) = \left(\frac{z^p}{(1-z)^{n+p}}\right)^{(\dagger)} * f(z) \quad (f \in \mathcal{A}_p). \tag{1.5}$$

If $f(z)$ is given by (1.1), then from (1.4) and (1.5), we deduce that

$$\begin{aligned} \mathcal{I}_{n,p} f(z) &= z^p + \sum_{k=p+1}^{\infty} \frac{(p+1)(p+2)\cdots k}{(n+p)(n+p+1)\cdots(n+k-1)} a_k z^k \\ &= z^p\, {}_2F_1(1, p+1; n+p; z) * f(z) \quad (n > -p). \end{aligned} \tag{1.6}$$

It follows from (1.6) that

$$z\big(\mathcal{I}_{n+1,p} f(z)\big)' = (n+p)\mathcal{I}_{n,p} f(z) - n\mathcal{I}_{n+1,p} f(z). \tag{1.7}$$

We also note that $\mathcal{I}_{0,p} f(z) = zf'(z)/p$ and $\mathcal{I}_{1,p} f(z) = f(z)$. Moreover, the operator $\mathcal{I}_{n,p} f$ defined by (1.5) is called as the Noor integral operator of $(n+p-1)$th order of f [6]. For $p = 1$, the operator $\mathcal{I}_{n,1} f \equiv \mathcal{I}_n f$ was introduced by Noor [10] and Noor and Noor [12]. Several classes of analytic functions, defined by using the operator $\mathcal{I}_n f$, have been studied by many authors [2,5,13]. More recently, Noor [11] introduced new subclasses of analytic functions associated with the Noor integral operator and studied their geometric properties. Now we define the following subclasses of analytic functions by using the operator $\mathcal{I}_{n,p}$.

For any integer n greater than $-p$ and arbitrary real numbers A, B $(-1 \leqslant B < A \leqslant 1)$, let $\mathcal{S}^*_{n,p}(\eta, A, B)$ be the class of functions $f \in \mathcal{A}_p$ satisfying the condition

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_{n,p} f(z))'}{\mathcal{I}_{n,p} f(z)} - \eta\right) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}) \tag{1.8}$$

for some $\eta (0 \leqslant \eta < p)$. We note that $\mathcal{S}^*_{1,p}(\eta, 1, -1)$ and $\mathcal{S}^*_{0,p}(\eta, 1, -1)$ are the classes of p-valently starlike and p-valently convex functions of order η in $\mathcal{U}$. Furthermore, $\mathcal{S}^*_{n,1}(\eta, A, B) = \mathcal{S}^*_n(\eta, A, B)$ is the class introduced and studied by Cho [2]. We denote $\mathcal{S}^*_{n,p}(\eta, 1, -1) \equiv \mathcal{S}^*_{n,p}(\eta)$. Let $H(p(z), zp'(z)) \prec h(z)$ be a first order differential subordination. Then a univalent function $q(z)$ is called its dominant if $p(z) \prec q(z)$ for all analytic functions $p(z)$ that satisfy the differential subordination. A dominant $\bar{q}(z)$ is called the best dominant if $\bar{q}(z) \prec q(z)$ for all dominants $q(z)$. For the general theory of first order differential subordination and its applications, we refer to [9].

In the present paper, we obtain inclusion relationships among the classes $\mathcal{S}^*_{n,p}(\eta, A, B)$. The integral preserving properties in connection with the operator $\mathcal{I}_{n,p} f$ are considered. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

2. Preliminaries

To establish our main results, we shall require the following lemmas.

Lemma 2.1 [8]. *If $-1 \leqslant B < A \leqslant 1, \beta > 0$ and the complex number γ satisfy $\Re(\gamma) \geqslant -\beta(1-A)/(1-B)$, then the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U})$$

has a univalent solution in $\mathcal{U}$ given by

$$q(z) = \begin{cases} \dfrac{z^{\beta+\gamma}(1+Bz)^{\beta(A-B)/B}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\beta(A-B)/B}\,dt} - \dfrac{\gamma}{\beta}, & B \neq 0, \\[2ex] \dfrac{z^{\beta+\gamma}\exp(\beta Az)}{\beta\int_0^z t^{\beta+\gamma-1}\exp(\beta At)\,dt} - \dfrac{\gamma}{\beta}, & B = 0. \end{cases} \tag{2.1}$$

If $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ *is analytic in* $\mathcal{U}$ *and satisfies*

$$\phi(z) + \frac{z\phi'(z)}{\beta\phi(z) + \gamma} \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}), \tag{2.2}$$

then

$$\phi(z) \prec q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U})$$

and $q(z)$ *is the best dominant of* (2.2).

Lemma 2.2 [16]. *Let* ν *be a positive measure on* $[0, 1]$. *Let* h *be a complex-valued function defined on* $\mathcal{U} \times [0, 1]$ *such that* $h(., t)$ *is analytic in* $\mathcal{U}$ *for each* $t \in [0, 1]$, *and* $h(z, .)$ *is* ν*-integrable on* $[0, 1]$ *for all* $z \in \mathcal{U}$. *In addition, suppose that* $\Re\{h(z, t)\} > 0$, $h(-r, t)$ *is real and* $\Re\{1/h(z, t)\} \geqslant 1/h(-r, t)$ *for* $|z| \leqslant r < 1$ *and* $t \in [0, 1]$. *If* $h(z) = \int_0^1 h(z, t)\, d\nu(t)$, *then* $\Re\{1/h(z)\} \geqslant 1/h(-r)$.

Each of the identities (asserted by Lemma 3 below) is well known [17].

Lemma 2.3. *For real numbers* a, b, c ($c \neq 0, -1, -2, \ldots$), *we have*

$$\int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}\, dt = \frac{\Gamma(b)\Gamma(c - b)}{\Gamma(c)}\, {}_2F_1(a, b; c; z) \quad (c > b > 0), \tag{2.3}$$

$${}_2F_1(a, b; c; z) = {}_2F_1(b, a; c; z), \tag{2.4}$$

$${}_2F_1(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1\left(a, c - b; c; \frac{z}{z - 1}\right). \tag{2.5}$$

3. Main results

We now prove

Theorem 3.1. *For any integer* n *greater than* $-p$ *and* $-1 \leqslant B < A \leqslant 1$, *if* $f \in \mathcal{S}^*_{n,p}(\eta, A, B)$, *then*

$$\frac{1}{p - \eta}\left\{\frac{z(\mathcal{I}_{n+1,p} f(z))'}{\mathcal{I}_{n+1,p} f(z)} - \eta\right\} \prec \frac{1}{p - \eta}\left\{\frac{1}{Q(z)} - (n + \eta)\right\} = q_1(z) \prec \frac{1 + Az}{1 + Bz} \quad (0 \leqslant \eta < p;\ z \in \mathcal{U}), \tag{3.1}$$

where

$$Q(z) = \begin{cases} \int_0^1 s^{p+n-1}\left(\frac{1+Bsz}{1+Bz}\right)^{\frac{(p-\eta)(A-B)}{B}} dt, & B \neq 0, \\ \int_0^1 s^{p+n-1} \exp((p - \eta)(s - 1)Az)\, dt, & B = 0, \end{cases} \tag{3.2}$$

$$q_1(z) = \frac{1}{p - \eta}\left\{\frac{p + n}{1 + Bz} - (n + \eta)\right\} \quad \text{when } A = -\frac{(n + \eta + 1)B}{p - \eta},\ B \neq 0,$$

and $q_1(z)$ is the best dominant of (3.1). *Furthermore, if $A \leqslant -(n+\eta+1)B/(p-\eta)$ with $-1 \leqslant B < 0$, then*

$$\mathcal{S}^*_{n,p}(\eta, A, B) \subset \mathcal{S}^*_{n+1,p}(\eta, 1-2\rho, -1), \tag{3.3}$$

where

$$\rho \equiv \rho(p, n, \eta, A, B) = \frac{1}{p-\eta}\left[(p+n)\left\{{}_2F_1\left(1, \frac{(p-\eta)(B-A)}{B}; p+n+1; \frac{B}{1-B}\right)\right\}^{-1} - (n+\eta)\right].$$

The result is best possible.

Proof. Let

$$\phi(z) = \frac{1}{p-\eta}\left\{\frac{z(\mathcal{I}_{n+1,p}f(z))'}{\mathcal{I}_{n+1,p}f(z)} - \eta\right\} \quad (z \in \mathcal{U}). \tag{3.4}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. By using the identity (1.7) in (3.4), we get

$$(n+p)\frac{\mathcal{I}_{n,p}f(z)}{\mathcal{I}_{n+1,p}f(z)} = (p-\eta)\phi(z) + n + \eta. \tag{3.5}$$

Taking the logarithmic derivatives in both sides of (3.5) and multiplying by z, we have

$$\frac{1}{p-\eta}\left\{\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)} - \eta\right\} = \phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+n+\eta} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{3.6}$$

Thus, $\phi(z)$ satisfies the differential subordination (2.2) and hence by using Lemma 2.1, we get

$$\phi(z) \prec q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

where $q_1(z)$ is given by (2.1) for $\beta = p - \eta$ and $\gamma = n + \eta$, and this $q_1(z)$ is the best dominant of (3.6). This proves the assertion (3.1).

Next we show that

$$\inf_{|z|<1}\left\{\Re\big(q(z)\big)\right\} = q(-1). \tag{3.7}$$

If we set $a = (p-\eta)(B-A)/B$, $b = p+n$ and $\gamma = p+n+1$ then $c > b > 0$. From (3.2), by using (2.4), (2.5) and (2.6), we see that for $B \neq 0$

$$Q(z) = (1+Bz)^a \int_0^1 s^{b-1}(1+Bsz)^{-a}\, ds = \frac{\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(1, a; c; \frac{Bz}{Bz+1}\right). \tag{3.8}$$

To prove (3.7), we need to show that $\Re\{1/Q(z)\} \geqslant 1/Q(-1)$, $z \in \mathcal{U}$. Since $A < -(n+\eta+1)B/(p-\eta)$ implies that $c > a > 0$, by using (2.4), (3.8) yields

$$Q(z) = \int_0^1 h(z,s)\, d\nu(s),$$

where

$$h(z,s)=\frac{1+Bz}{1+(1-s)Bz}\quad(0\leqslant s\leqslant 1)\quad\text{and}$$
$$d\nu(s)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}s^{a-1}(1-s)^{c-a-1}\,ds,$$

which is a positive measure on [0,1]. For $-1\leqslant B<0$, it may be noted that $\Re\{h(z,s)\}>0$, $h(-r,s)$ is real for $0\leqslant r<1$, $0\in[0,1]$ and

$$\Re\left\{\frac{1}{h(z,s)}\right\}=\Re\left\{\frac{1+(1-s)Bz}{1+Bz}\right\}\geqslant\frac{1-(1-s)Br}{1-Br}=\frac{1}{h(-r,s)}$$

for $|z|\leqslant r<1$ and $s\in[0,1]$. Therefore, by using Lemma 2.2, we have $\Re\{1/Q(z)\}\geqslant 1/Q(-r)$, $|z|\leqslant r<1$ and by letting $r\to 1^-$, we obtain $\Re\{1/Q(z)\}\geqslant 1/Q(-1)$. Further, by taking $A\to(-(n+\eta+1)B/(p-\eta))^+$ for the case $A=-(n+\eta+1)B/(p-\eta)$, and using (3.1) we get (3.3).

The result is best possible as the function $q_1(z)$ is the best dominant of (3.1). This completes the proof of Theorem 1. □

Setting $A=1$ and $B=-1$ in Theorem 3.1, we deduce that

Corollary 3.1. *For any integer n greater than $-p$ and $(p-n-1)/2\leqslant\eta<p$, we have*

$$\mathcal{S}^*_{n,p}(\eta)\subset\mathcal{S}^*_{n+1,p}(\xi),$$

where

$$\xi\equiv\xi(p,n,\eta)=(p+n)\left\{{}_2F_1\big(1,2(p-\eta);p+n+1;1/2\big)\right\}^{-1}-n.$$

The result is best possible.

Putting $n=0$ in Corollary 3.1, we get

Corollary 3.2. *For $(p-1)/2\leqslant\eta<p$, we have*

$$\mathcal{K}_p(\eta)\subset\mathcal{S}^*_p(\varkappa),$$

where $\varkappa\equiv\varkappa(p,\eta)=p\{{}_2F_1(1,2(p-\eta);p+1;1/2)\}^{-1}$. The result is best possible.

Remark 3.1.

(i) From (3.1), it is easily seen that Theorem 3.1 improves the corresponding result of Cho [2] for $p=1$.

(ii) Noting that

$$\left\{{}_2F_1\big(1,2(1-\eta);2;1/2\big)\right\}^{-1}=\begin{cases}\frac{1-2\eta}{2^{2(1-\eta)}(1-2^{2\eta-1})}, & \eta\neq\frac{1}{2},\\ \frac{1}{2\ln 2}, & \eta=\frac{1}{2},\end{cases}$$

Corollary 3.2 yields the corresponding result due to MacGregor [7] (see also [14]) for $p=1$.

(iii) It is proved [14] that if $p \geqslant 2$ and $f \in \mathcal{K}_p(0)$, then f is p-valently starlike in $\mathcal{U}$ but is not necessarily p-valently starlike of order larger than zero in $\mathcal{U}$. However, our Corollary 3.2 shows that if f is p-valently convex of order at least $(p-1)/2$, then f is p-valently starlike of order larger than zero in $\mathcal{U}$.

Theorem 3.2. *If $f \in \mathcal{S}^*_{n+1,p}(\eta)$ for any integer n greater than $-p$ and η $(0 \leqslant \eta < p)$, then $f \in \mathcal{S}^*_{n,p}(\eta)$ for $|z| < R(p,n,\eta)$, where*

$$R(p,n,\eta) = \begin{cases} \frac{(p-\eta+1)-\sqrt{(p-\eta+1)^2-(p+n)(p-2\eta-n)}}{p-2\eta-n}, & \eta \neq \frac{p-n}{2}, \\ \frac{p+n}{p+n+2}, & \eta = \frac{p-n}{2}. \end{cases} \tag{3.9}$$

The bound $R(p,n,\eta)$ is best possible.

Proof. We have

$$\frac{z(\mathcal{I}_{n+1,p}f(z))'}{\mathcal{I}_{n+1,p}f(z)} = \eta + (p-\eta)u(z), \tag{3.10}$$

where $u(z) = 1 + u_1 z + u_2 z^2 + \cdots$ is analytic and has a positive real part in $\mathcal{U}$. Using the identity (1.7) in (3.10) and taking logarithmic differentiation in the resulting equation, we deduce that

$$\begin{aligned} \Re\left\{\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)}\right\} - \eta &= (p-\eta)\Re\left\{u(z) + \frac{zu'(z)}{n+\eta+(p-\eta)u(z)}\right\} \\ &= (p-\eta)\Re\left\{u(z) - \frac{|zu'(z)|}{|n+\eta+(p-\eta)u(z)|}\right\}. \end{aligned} \tag{3.11}$$

Now by using the well-known estimates [7]

$$\left|zu'(z)\right| \leqslant \frac{2r}{1-r^2}\Re\left\{u(z)\right\} \quad \text{and} \quad \Re\left\{u(z)\right\} \geqslant \frac{1-r}{1+r} \quad \left(|z| = r < 1\right)$$

in (3.11), we get

$$\begin{aligned} \Re\left\{\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)}\right\} - \eta &\geqslant (p-\eta)\Re\left\{u(z)\right\} \\ &\quad \times \left\{1 - \frac{2r}{(n+\eta)(1-r^2)+(p-\eta)(1-r)^2}\right\}, \end{aligned}$$

which is certainly positive if $r < R(p,n,\eta)$, where $R(p,n,\eta)$ is given by (3.9).

To show that the bound $R(p,n,\eta)$ is best possible, we consider the function $f \in \mathcal{A}_p$ defined in $\mathcal{U}$ by

$$\frac{z(\mathcal{I}_{n+1,p}f(z))'}{\mathcal{I}_{n+1,p}f(z)} = \eta + (p-\eta)\frac{1+z}{1-z} \quad (0 \leqslant \eta < p;\ n > -p).$$

Noting that

$$\begin{aligned} \Re\left\{\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)}\right\} - \eta &= (p-\eta)\Re\left\{\frac{1+z}{1-z} + \frac{2z}{(n+\eta)(1-z^2)+(p-\eta)(1+z)^2}\right\} \\ &= 0 \end{aligned}$$

for $z = -R(p, n, \eta)$, we conclude that the bound is best possible. This proves Theorem 3.2. □

For $n = 0$, Theorem 3.2 yields

Corollary 3.3. *If $f \in \mathcal{S}_p^*(\eta)$ $(0 \leqslant \eta < p)$, then $f \in K_p(\eta)$ for $|z| < \xi(p, \eta)$, where*

$$\xi(p,\eta) = \begin{cases} \frac{(p-\eta+1)-\sqrt{\eta^2+2(p-\eta)+1}}{p-2\eta}, & \alpha \neq \frac{p}{2}, \\ \frac{p}{p+2}, & \alpha = \frac{p}{2}. \end{cases}$$

The bound $\xi(p, \eta)$ is best possible.

For a function $f \in \mathcal{A}_p$, given by (1.1), the integral operator $\mathcal{F}_{\delta,p}$ [4] is defined by

$$\begin{aligned} F_{\delta,p}(f) = F_{\delta,p}(f)(z) &= \frac{\delta+p}{z^\delta} \int_0^z t^{\delta-1} f(t)\,dt \\ &= z^p + \sum_{k=1}^{\infty} \frac{\delta+p}{\delta+p+k} a_{p+k} z^{p+k} \quad (\delta > -p;\ z \in \mathcal{U}). \end{aligned} \tag{3.12}$$

We denote $F_{\delta,1}(f)(z) = F_\delta(f)(z)$. It readily follows from (3.12) that

$$f \in \mathcal{A}_p \quad \Longrightarrow \quad \mathcal{F}_{\delta,p} \in \mathcal{A}_p.$$

Theorem 3.3. *Let n be any integer greater than $-p$, $-1 \leqslant B < A \leqslant 1$ and δ be a real number satisfying*

$$\delta \geqslant -\eta - \frac{(p-\eta)(1-A)}{(1-B)}. \tag{3.13}$$

(i) *If $f \in \mathcal{S}^*_{n,p}(\eta, A, B)$, then the function $\mathcal{F}_{\delta,p}$ defined by* (3.12) *belongs to $\mathcal{S}^*_{n,p}(\eta, A, B)$. Furthermore, we have*

$$\begin{aligned} \frac{1}{p-\eta}\left\{\frac{z(\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z))'}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z)} - \eta\right\} &\prec \frac{1}{p-\eta}\left\{\frac{1}{Q(z)} - (\delta+\eta)\right\} \\ &= q_2(z) \prec \frac{1+Az}{1+Bz} \quad (0 \leqslant \eta < p;\ z \in \mathcal{U}), \end{aligned} \tag{3.14}$$

where

$$Q(z) = \begin{cases} \int_0^1 s^{p+\delta-1}\left(\frac{1+Bsz}{1+Bz}\right)^{\frac{(p-\eta)(A-B)}{B}} dt, & B \neq 0, \\ \int_0^1 s^{p+\delta-1} \exp((p-\eta)(s-1)Az)\,dt, & B = 0. \end{cases} \tag{3.15}$$

and $q_2(z)$ is the best dominant of (3.14).

(ii) *If $-1 \leqslant B < 0$ and $\delta \geqslant \max\left\{\frac{(p-\eta)(B-A)}{B} - p - 1, -\frac{(p-\eta)(1-A)}{1-B} - \eta\right\}$, then for $f \in \mathcal{S}^*_{n,p}(\eta, A, B)$, we have $\mathcal{F}_{\delta,p}(f) \in \mathcal{S}^*_{n,p}(\eta, 1-2\kappa, -1)$, where*

$$\kappa \equiv \kappa(p, n, \eta, A, B)$$
$$= \frac{1}{p-\eta}\left[(p+\delta)\left\{{}_2F_1\left(1, \frac{(p-\eta)(B-A)}{B}; p+\delta+1; \frac{B}{1-B}\right)\right\}^{-1} - (\delta+\eta)\right].$$

The result is best possible.

Proof. From (1.6) and (3.12), we have

$$z\big(\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z)\big)' = (p+\delta)\mathcal{I}_{n,p}f(z) - \delta\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z). \tag{3.16}$$

Let

$$\phi(z) = \frac{1}{p-\eta}\left\{\frac{z(\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z))'}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z)} - \eta\right\}. \tag{3.17}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. By using (3.16) in (3.17), we deduce that

$$(p+\delta)\frac{\mathcal{I}_{n,p}f(z)}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p}(f)(z)} = (p-\eta)\phi(z) + \delta + \eta. \tag{3.18}$$

Taking logarithmic derivatives in both sides of (3.18) and multiplying by z, we have

$$\frac{1}{p-\eta}\left\{\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)} - \eta\right\} = \phi(z) + \frac{z\phi'(z)}{(p-\eta)\phi(z)+\delta+\eta} \quad (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we obtain

$$\phi(z) \prec q_2(z) = \frac{1}{p-\eta}\left\{\frac{1}{Q(z)} - (\delta+\eta)\right\} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

where $Q(z)$ is given by (3.15). This proves the first part of the theorem.

Moreover, the second part follows from the similar arguments as in the proof of Theorem 3.1 and so we omit the details. □

Taking $A = 1$ and $B = -1$ in Theorem 3.3, we obtain

Corollary 3.4. *If δ is a real number satisfying $\delta \geqslant \max\{2(p-\eta) - p - 1, -\eta\}$ and $f \in \mathcal{S}^*_{n,p}(\eta)$, then $\mathcal{F}_{\delta,p}(f)(z) \in \mathcal{S}^*_{n,p}(\tau)$, where*

$$\tau \equiv \tau(p, \delta, \eta) = (p+\delta)\big\{{}_2F_1\big(1, 2(p-\eta); p+\delta+1; 1/2\big)\big\}^{-1} - \delta.$$

The result is best possible.

Remark 3.2.

(i) For $p = 1$, Theorem 3.3 improves the corresponding result of Cho [2].

(ii) Substituting $n = p = 1$ (or $n = 0$ and $p = 1$ respectively) in Corollary 3.4, we observe that for $0 \leqslant \eta < 1$ and $\delta \geqslant -\eta$,

$$f \in \mathcal{S}^*(\eta) \quad \Longrightarrow \quad \mathcal{F}_\delta(f) \in S^*\left(\frac{\delta+1}{{}_2F_1(1, 2(1-\eta); \delta+2; 1/2)} - \delta\right),$$

$$f \in \mathcal{K}(\eta) \quad \Longrightarrow \quad \mathcal{F}_\delta(f) \in \mathcal{K}\left(\frac{\delta+1}{{}_2F_1(1, 2(1-\eta); \delta+2; 1/2)} - \delta\right).$$

Further, from Corollary 3.4 and the well-known result $\mathcal{K}(0) \equiv \mathcal{K} \subset \mathcal{S}^*(1/2)$, we obtain the following:

$$\mathcal{F}_\delta(\mathcal{K}) \subset \mathcal{F}_\delta\left(\mathcal{S}^*\left(\frac{1}{2}\right)\right) \subset \mathcal{S}^*\left(\frac{\delta+1}{{}_2F_1(1, 1; \delta+2; 1/2)} - \delta\right).$$

In the special case,

$$\mathcal{F}_0(\mathcal{K}) \subset \mathcal{F}_0\left(\mathcal{S}^*\left(\frac{1}{2}\right)\right) \subset \mathcal{S}^*\left(\frac{1}{2\ln 2}\right) \quad \text{and}$$

$$\mathcal{F}_1(\mathcal{K}) \subset \mathcal{F}_1\left(\mathcal{S}^*\left(\frac{1}{2}\right)\right) \subset \mathcal{S}^*\left(\frac{2\ln 2 - 1}{2(1-\ln 2)}\right).$$

All the above results are best possible and they improve the corresponding work of Fukui et al. [3].

Theorem 3.4. *Let n be any integer greater than $-p$, $-1 \leqslant B < A \leqslant 1$ and δ be a real number satisfying* (3.13).

(i) *If $f \in \mathcal{S}^*_{n,p}(\eta, A, B)$, then the function $\mathcal{F}_{\delta,p}$ defined by* (3.12) *satisfies*

$$\frac{\mathcal{I}_{n,p} f(z)}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p} f(z)} \prec \frac{1}{(\delta+p)Q(z)} \equiv q_3(z) \prec \frac{1+A'z}{1+Bz} \quad (z \in \mathcal{U}), \tag{3.19}$$

where $A' = \{(p-\eta)A + (\delta+\eta)B\}/(p+\delta)$, $Q(z)$ is given by (3.15), *and $q_3(z)$ is the best dominant of* (3.19).

(ii) *Furthermore, if $-1 \leqslant B < 0$, $B < A \leqslant \min\{1 + \frac{(p+\delta)(1-B)}{p-\eta}, -\frac{(\eta+p+1)B}{p-\eta}\}$, and $f \in \mathcal{S}^*_{n,p}(\eta, A, B)$, then*

$$\Re\left\{\frac{\mathcal{I}_{n,p} f(z)}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p} f(z)}\right\} > \frac{1}{{}_2F_1\left(1, \frac{(p-\eta)(B-A)}{B}; \delta+p+1; \frac{B}{(B-1)}\right)} \quad (z \in \mathcal{U}).$$

The result is best possible.

Proof. Setting

$$\phi(z) = \frac{\mathcal{I}_{n,p} f(z)}{\mathcal{I}_{n,p}\mathcal{F}_{\delta,p} f(z)} \quad (z \in \mathcal{U}), \tag{3.20}$$

we see that $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. Making use of the logarithmic differentiation in (3.20) and using the identity (3.16) in the resulting equation, we get

$$\frac{1}{p-\eta}\left(\frac{z(\mathcal{I}_{n,p}f(z))'}{\mathcal{I}_{n,p}f(z)}-\eta\right)=P(z)+\frac{zP'(z)}{(p-\eta)P(z)+(\delta+\eta)}$$
$$\prec\frac{1+Az}{1+Bz}\quad(z\in\mathcal{U}),\tag{3.21}$$

where $P(z)=\{(\delta+p)\phi(z)-(\delta+\eta)\}/(p-\eta)$. By using Lemma 3.1, we have

$$P(z)\prec q_3(z)\prec\frac{1+Az}{1+Bz}\quad(z\in\mathcal{U}),\tag{3.22}$$

where $q_3(z)$ is the best dominant of (3.21), $q_3(z)$ is given by (2.1) with $\beta=p-\eta$ and $\gamma=\delta+\eta$. Again, by using (3.21) in (3.22), we get (3.19).

The remaining part of proof of Theorem 3.4 is similar to that of Theorem 3.1, and so we omit the details. □

By taking $n=A=1$ and $B=-1$ in Theorem 3.4, we have

Corollary 3.5. *If* $f\in\mathcal{S}_p^*(\eta)$, $\max\{-\delta,(p-\delta-1)/2\}\leqslant\eta<p$, *and* $\delta+p>0$, *then*

$$\Re\left\{\frac{z^\delta f(z)}{\int_0^z t^{\delta-1}f(t)\,dt}\right\}>\varrho(p,\delta,\eta)\quad(z\in\mathcal{U}),$$

where $\varrho(p,\delta,\eta)=(p+\delta)[{}_2F_1(1,2(p-\eta);\delta+p+1;1/2)]^{-1}$. *The result is best possible.*

For $n=0$, $A=1$ and $B=-1$, Theorem 3.4 yields the following result.

Corollary 3.6. *If* $f\in\mathcal{K}_p(\eta)$, $\max\{-\delta,(p-\delta-1)/2\}\leqslant\eta<p$, *and* $\delta+p>0$, *then*

$$\Re\left\{\frac{zf'(z)}{f(z)-\frac{\delta}{z^\delta}\int_0^z t^{\delta-1}f(t)\,dt}\right\}>\varrho(p,\delta,\eta)\quad(z\in\mathcal{U}),$$

where $\varrho(p,\delta,\eta)$ *is defined as in Corollary* 3.5. *The result is best possible.*

Remark 3.3. We note that Corollaries 3.5 and 3.6 improve the corresponding of Aouf et al. [1].

Acknowledgments

The authors express their gratitude to the reviewer(s) for suggesting many improvements.

References

[1] M.K. Aouf, H.M. Hossen, H.M. Srivastava, Some families of multivalent functions, Comput. Math. Appl. 39 (2000) 39–48.
[2] N.E. Cho, The Noor integral operator and strongly close-to-convex functions, J. Math. Anal. Appl. 283 (2003) 202–212.

[3] S. Fukui, J.A. Kim, H.M. Srivastava, On certain subclass of univalent functions by some integral operators, Math. Japon. 50 (1999) 359–370.
[4] R.M. Goel, N.S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc. 78 (1980) 353–357.
[5] J.-L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261 (2001) 441–447.
[6] J.-L. Liu, K.I. Noor, Some properties of Noor integral operator, J. Natur. Geom. 21 (2002) 81–90.
[7] T.H. MacGregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14 (1963) 514–520.
[8] S.S. Miller, P.T. Mocanu, Univalent solutions of Briot–Bouquet differential subordinations, J. Differential Equations 58 (1985) 297–309.
[9] S.S. Miller, P.T. Mocanu, Differential Subordinations: Theory and Applications, Pure Appl. Math., vol. 225, Dekker, New York, 2000.
[10] K.I. Noor, On new classes of integral operators, J. Natur. Geom. 16 (1999) 71–80.
[11] K.I. Noor, Some classes of p-valent analytic functions defined by certain integral operator, Appl. Math. Comput. 157 (2004) 835–840.
[12] K.I. Noor, M.A. Noor, On integral operators, J. Math. Anal. Appl. 238 (1999) 341–352.
[13] K.I. Noor, M.A. Noor, On certain classes of analytic functions defined by Noor integral operator, J. Math. Anal. Appl. 281 (2003) 244–252.
[14] M. Nunokawa, On the theory of univalent functions, Tsukuba J. Math. 11 (1987) 273–286.
[15] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc. 49 (1975) 109–115.
[16] D.R. Wilken, J. Feng, A remark on convex and starlike functions, J. London Math. Soc. 21 (1980) 287–290.
[17] E.T. Whittaker, G.N. Watson, A Course on Modern Analysis, fourth ed. (reprinted), Cambridge Univ. Press, Cambridge, 1927.